A Nonlinear Singular Diffusion Equation with Source

Pan Jiaqing
Email:jqpan@jmu.edu.cn
Department of Mathematics, Jimei University,
Xiamen 361021, P.R.China

Abstract: In this paper, the existence, uniqueness and dependence on initial value of solution for a singular diffusion equation with nonlinear boundary condition are discussed. It is proved that there exists a unique global smooth solution which depends on initial data continuously.

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1 Introduction

In this paper, we consider a boundary value problem

$$\begin{cases} u_t = (u^{m-1}u_x)_x + u^p, & 0 < x < 1, t > 0, \\ u_x|_{x=0} = 0, \qquad u_x|_{x=1} = -u^\alpha, & t \geq 0, \\ u|_{t=0} = u_0, & 0 \leq x \leq 1. \end{cases} \tag{1.1}$$

Where $-1 < m < 0,\ 0 < p < 1,\ 2 - m < \alpha$ and $0 \leq u_0(x) \leq M,\ \int_0^1 u_0(x)dx > 0$.

The equation in (1.1) arises in many applications in physics and chemistry . For example, it has been proposed for $m = \frac{1}{2}$ in plasma physics ([8]), and for $m = -1$ in the heat conduction in solid hydrogen ([7]).

Although there are many results for $m > 0$, the situation is completely different for $m < 0$, where the equation becomes singular since u^m blows up as $u \longrightarrow 0$ and $\int_0^t \int_0^1 u^m dx d\tau$ can be unbounded. Thus there is essential singularity in (1.1) when $u = 0$. Some authors have discussed the similar problems with $u_0 > 0$. For example, for positive initial value u_0, H. Zhang ([11]) discussed the Cauchy problem for $m \in (-1, 0]$ with the conditions

$$\lim_{x \longrightarrow -\infty} u^{m-1}u_x = \lambda, \qquad \lim_{x \longrightarrow +\infty} u = 1.$$

Where, $\lambda > 0$. The author also discussed the first boundary value problem for $-1 < m < 0$ but $u_0 \geq 0$ ([12]). In order to obtain our conclusions of the paper, we divide the range $[0, +\infty)$ into two parts: $[0, t_*]$ and $[t_*, +\infty)$. We first use Arzela's theorem to prove that there exists a function u^* which solves (1.1) on $[0, 1] \times [0, t_*]$. Notice that $u^*(x, t_*) > 0$

and $u^*(x,t_*)$ is smooth, so we use $u^*(x,t_*)$ as a new initial value and then obtain another solution u^{**} on $[0,1]\times[t_*,+\infty)$. Thus we obtain a solution

$$u(x,t)=\begin{cases} u^*(x,t), & t\in[0,t_*],\\ u^{**}(x,t), & t\in[t_*,+\infty). \end{cases}$$

Finally, with a comparison theorem, we can prove the uniqueness and the continuous dependence on initial value.

By a solution of (1.1), we mean a function $u(x,t)$ is smooth enough and satisfies the equation in (1.1), u_x is continuous up to $x=0,1$ and satisfies the boundary condition of (1.1) and $\lim\limits_{t\longrightarrow 0}\int_0^1|u-u_0|dx=0$.

The following notations will be used throughout the paper:

$$G_T=(0,1)\times(0,T),\qquad G=(0,1)\times(0,+\infty),\qquad \overline{u}_0=\int_0^1 u_0dx.$$

The main results of our paper are as follows:

Theorem Assume

$$-1<m<0,\ 0<p<1,\ 2-m<\alpha,\ 0\le u_0(x)\le M,\ \overline{u}_0>0. \tag{1.2}$$

Then there exists a unique global smooth positive solution $u(x,t)$ to the problem (1.1) such that

$$u\in C^\infty(G)\cap C([0,+\infty);L^1(0,1)).$$

If $u,\hat{u}$ are two solutions corresponding to $u_0,\hat{u}_0$, then for any $T>0$, there is a positive constant C such that

$$\int_0^1|u-\hat{u}|dx\le C\int_0^1|u_0-\hat{u}_0|dx,\qquad \text{for } t\in[0,T]. \tag{1.3}$$

2 Preliminary lemmas

Lemma 1 Assume $0<u_0\le M$ and u_0 be smooth enough. For any $T>0$, if $u(x,t)$ is a smooth positive solution to the problem (1.1) on G_T, then there exists a positive constant $C_0>0$ such that

$$\|u\|_{L^\infty(G_T)}\le C_0,$$

where,

$$C_0=[(1-p)T+M^{1-p}]^{\frac{1}{1-p}}.$$

Proof: For any $q \geq 0$, we have

$$u^q u_t = u^q (u^{m-1} u_x)_x + u^{p+q}.$$

By Holder's inequality,

$$\begin{aligned} \frac{1}{1+q}\frac{d}{dt}\int_0^1 u^{q+1}dx &\leq \int_0^1 u^{p+q}dx \\ &\leq (\int_0^1 u^{1+q}dx)^{\frac{p+q}{1+q}}. \end{aligned} \tag{2.1}$$

So,

$$\begin{aligned} \frac{d}{dt}(\int_0^1 u^{1+q}dx)^{\frac{1-p}{1+q}} &\leq 1-p \\ \|u\|_{L^{1+q}(0,1)} &\leq [(1-p)t + \|u_0\|^{1-p}_{L^{1+q}(0,1)}]^{\frac{1}{1-p}} \\ &\leq C_0, \qquad \text{for } t \in [0,T],\ q \geq 0. \end{aligned} \tag{2.2}$$

By [10](Th 2.8, p.25), $\|u\|_{L^\infty(G_T)} \leq C_0$.

Lemma 2 Assume u_0 and $u(x,t)$ be as lemma 1, then

$$|(u^{\frac{m}{q}})_x| \leq C_T(1 + t^{-\frac{1}{2}}), \qquad \text{for } (x,t) \in [0,1] \times (0,T).$$

Where, $q = \frac{3m-1}{2(m-1)}$, C_T depends on T, m, M, p and α.

Proof: Set $u^m = V^q$, then

$$V_t = V^{q-\frac{q}{m}}V_{xx} + (q-1)V^{q-1-\frac{q}{m}}(V_x)^2 + \frac{m}{q}V^{1+\frac{qp-q}{m}}.$$

Differentiating this equation with respect to x and then multiplying through by V_x, letting $V_x = h$, yields

$$\begin{aligned} \frac{1}{2}(h^2)_t - V^{q-\frac{q}{m}}hh_{xx} &= (3q-2-\frac{q}{m})V^{q-1-\frac{q}{m}}h^2h_x + \frac{m}{q}(1+\frac{pq-q}{m})V^{\frac{pq-q}{m}}h^2 \\ &\quad +(q-1)(q-1-\frac{q}{m})V^{q-2-\frac{q}{m}}h^4. \end{aligned} \tag{2.3}$$

For any $0 < \tau < T$, let $\phi(t)$ be a smooth function and

$$\phi(t) = \begin{cases} 0, & t \leq 0, \\ \text{monotone}, & 0 < t < \tau, \\ 1, & t \geq \tau. \end{cases}$$

Thus there is a positive constant $C_* > 0$ such that $0 \le \frac{d\phi}{dt} \le \frac{C_*}{\tau}$. Set $Z = (\phi h)^2$. By [1](Th.6, p.65), we have $Z \in C(\overline{G}_T)$. Clearly, $Z|_{t=0} = Z|_{x=0} = 0$ and (since $\frac{m}{q} - 1 + \alpha > 0$)

$$Z|_{x=1} \le (\frac{m}{q})^2 C_0^{2(\frac{m}{q}-1+\alpha)}. \tag{2.4}$$

Let

$$Z(x_0, t_0) = \max_{(x,t)\in\overline{G}_T} Z(x,t),$$

if $0 < x_0 < 1$ and $t_0 > 0$, then

$$Z_t \ge 0, \quad Z_x = 0, \quad Z_t - V^{q-\frac{q}{m}} Z_{xx} \ge 0, \qquad \text{at } (x_0, t_0).$$

Hence,

$$-\phi\phi_t h^2 \le \phi^2[\frac{1}{2}(h^2)_t - V^{q-\frac{q}{m}} h h_{xx}], \qquad \text{at } (x_0, t_0).$$

Multiplying (2.3) by ϕ^2, we have

$$(1-q)(q-1-\frac{q}{m})Z \le \frac{m}{q}(1+\frac{pq-q}{m})u^{p-m+\frac{2m}{q}}\phi^2 + \frac{C_*}{\tau}u^{\frac{2m}{q}+1-m}, \qquad \text{at } (x_0, t_0).$$

Since $p < 1, m \in (-1, 0)$ and $q > 0$, thus $\frac{m}{q}(1 + \frac{pq-q}{m}) < 0$. Thus we have

$$(1-q)(q-1-\frac{q}{m})Z \le \frac{C_*}{\tau}u^{\frac{2m}{q}+1-m}, \qquad \text{at } (x_0, t_0).$$

Notice that $q = \frac{3m-1}{2(m-1)}$, hence

$$(1-q)(q-1-\frac{q}{m}) > 0, \qquad \frac{2m}{q} + 1 - m > 0.$$

Let

$$C^{**} = \frac{C_* C_0^{\frac{2m}{q}+1-q}}{(1-q)(q-1-\frac{q}{m})}.$$

Thus,

$$\begin{aligned} Z(x_0, t_0) &\le \frac{C_* C_0^{\frac{2m}{q}+1-q}}{\tau(1-q)(q-1-\frac{q}{m})} \\ &= \frac{C^{**}}{\tau}. \end{aligned} \tag{2.5}$$

Recall from $Z(x_0, t_0)$ that (2.5) holds for all $(x,t) \in (0,1)\times(0,T)$, specially, for $0 < x < 1,\ t = \tau$ (here, $\phi = 1, Z = h^2(x,\tau)$), thus

$$|h(x,\tau)| \le (\frac{C^{**}}{\tau})^{\frac{1}{2}}, \qquad \text{for } (x,\tau) \in (0,1)\times(0,T).$$

By (2.4), there is another positive constant C_T which depends on T, m, M, p and α such that

$$\begin{aligned} |h(x,\tau)| &\le |\frac{m}{q}|C_0^{(\frac{m}{q}-1+\alpha)} + (\frac{C^{**}}{\tau})^{\frac{1}{2}} \\ &\le C_T(1+\tau^{-\frac{1}{2}}), \qquad \text{for } (x,\tau) \in [0,1]\times(0,T). \end{aligned}$$

The proof is complete.

We notice that C_T increases with respect to C_0 by (2.5) and C_0 increases with respect to T by lemma 1. So we have

Corollary If $T_1 \le T_2$, then $C_{T_1} \le C_{T_2}$.

Lemma 3 Assume $u_0(x)$ and $u(x,t)$ be as lemma 1, then

$$\int_0^1 u(x,t)dx \ge [(\alpha+m-2)t + \int_0^1 u_0^{2-m-\alpha}dx]^{\frac{1}{1-m-\alpha}}, \qquad \text{for } t \in [0,T].$$

Proof: Multiplying $u^{1-m-\alpha}$ to the equation in (1.1) yields

$$\frac{1}{2-m-\alpha}(u^{2-m-\alpha})_t = \frac{1}{m}u^{1-m-\alpha}(u^m)_{xx} + u^{p+1-m-\alpha}.$$

Because of $2-m < \alpha$ and $u(x,t) > 0$, thus

$$\begin{aligned} \frac{d}{dt}\int_0^1 u^{2-m-\alpha}dx &= (2-m-\alpha)[(m-1+\alpha)\int_0^1 u^{-1-\alpha}(u_x)^2dx \\ &\quad + \int_0^1 u^{p+1-m-\alpha}dx - 1] \\ &\le \alpha+m-2, \end{aligned}$$

so

$$\int_0^1 u^{2-m-\alpha}dx \le \int_0^1 u_0^{2-m-\alpha}dx + (\alpha+m-2)t. \tag{2.6}$$

By Hölder's inverse-inequality([10], Ch.2, Th.2.6), we have

$$(\int_0^1 udx)^{2-m-\alpha} \le \int_0^1 u^{2-m-\alpha}dx.$$

Hence by (2.6), we have

$$\int_0^1 u(x,t)dx \ \geq \ [(\alpha+m-2)t+\int_0^1 u_0^{2-m-\alpha}dx]^{\frac{1}{2-m-\alpha}}.$$

Lemma 4 Assume $u_1, u_2 \in C([0,T], L^1(0,1))$ be two solutions corresponding to u_{10} and u_{20}, then

$$\int_0^1 |u_2-u_1|dx \leq \int_0^1 |u_{20}-u_{10}|dx + \int_0^1\int_0^t |u_2^p-u_1^p|dxd\tau, \qquad \text{for } t\in[0,T].$$

Proof: Take a function $p(x)\in C^\infty(R)$ such that

$$p(x) = \begin{cases} 0, & x\leq 0, \\ \exp[\frac{-1}{x^2}\exp\frac{-1}{(x-1)^2}], & 0<x<1, \\ 1, & x\geq 1. \end{cases}$$

Clearly, $0\leq p(x)\leq 1$ and $p'(x)\geq 0$. For any given $\varepsilon>0$, let $p_\varepsilon(x)=p(\frac{x}{\varepsilon})$. Set

$$w=\frac{1}{m}(u_2^m-u_1^m).$$

Then $w>0$ iff $u_2>u_1$. Thus

$$\begin{aligned} \int_0^1 (u_2-u_1)_t p_\varepsilon(w)dx &= \int_0^1 w_{xx}p_\varepsilon(w)dx + \int_0^1 (u_2^p-u_1^p)p_\varepsilon(w)dx \\ &\leq (u_1^{m-1+\alpha}-u_2^{m-1+\alpha})p_\varepsilon(w)|_{x=1} + \int_0^1 (u_2^p-u_1^p)p_\varepsilon(w)dx. \end{aligned}$$

If $u_2(1,t)>u_1(1,t)$, then $(u_1^{m-1+\alpha}-u_2^{m-1+\alpha})p_\varepsilon(w)|_{x=1}<0$ (owing to $\alpha>2-m$). If $u_2(1,t)\leq u_1(1,t)$, then $w|_{x=1}\leq 0$ and therefore, $p_\varepsilon(w)|_{x=1}=0$. Thus we always have $(u_1^{m-1+\alpha}-u_2^{m-1+\alpha})p_\varepsilon(w)|_{x=1}\leq 0$ and

$$\int_0^1 (u_2-u_1)_t p_\varepsilon(w)dx \leq \int_0^1 (u_2^p-u_1^p)p_\varepsilon(w)dx. \tag{2.7}$$

Since lemma 3.1 of [9] shows

$$\int_0^1 (u-\hat{u})_t p_\varepsilon(w)dx \longrightarrow \frac{d}{dt}\int_0^1 [u-\hat{u}]_+dx, \qquad \text{as } \varepsilon\longrightarrow 0,$$

thus,

$$\int_0^1 [u_2-u_1]_+dx \leq \int_0^1 [u_{20}-u_{10}]_+dx + \int_0^1\int_0^t [u_2^p-u_1^p]_+dxd\tau, \qquad \text{for } t\in[0,T],$$

$$(2.8)$$

in which, $[u-\hat{u}]_+ = \max(u-\hat{u},0)$. Similarly,

$$\int_0^1 [u_2-u_1]_- dx \le \int_0^1 [u_{20}-u_{10}]_- dx + \int_0^1\int_0^t [u_2^p-u_1^p]_- dx d\tau, \qquad \text{for } t\in[0,T], \tag{2.9}$$

where, $[u-\hat{u}]_- = -\min(u-\hat{u},0)$. By (2.8) and (2.9), we know that the lemma is true.

3 Proof of the Theorem

We prove our theorem by two steps.

STEP 1 In this step, we assume that $0 < u_0 \le M$ and u_0 is smooth enough, $u_{0x}|_{x=0} = 0$, $(u_{0x}+u_0^\alpha)|_{x=1} = 0$. We will prove that there exists a unique global smooth solution of (1.1).

For any given $T>0$, we consider the problem (1.1) on $\overline{G}_T$. Make two smooth functions as the following ([9], p.997):

$$h(r) = \begin{cases} \frac{1}{2}(2\overline{M})^{m-1}, & r \ge 2\overline{M}, \\ \text{monotone}, & \overline{M} < r < 2\overline{M}, \\ r^{m-1}, & \delta \le r \le \overline{M}, \\ \text{monotone}, & 0 \le r < \delta, \\ 2\delta^{m-1}, & r < 0. \end{cases}$$

$$g(r) = \begin{cases} \frac{1}{2}(2\overline{M})^{m-2}, & r \ge 2\overline{M}, \\ \text{monotone}, & \overline{M} < r < 2\overline{M}, \\ r^{m-2}, & \delta \le r \le \overline{M}, \\ \text{monotone} & 0 \le r < \delta, \\ 2\delta^{m-2} f(r), & r < 0. \end{cases}$$

Where, $0 < \delta < \min\limits_{x\in[0,1]} u_0(x)$, $\overline{M} > M$. $\overline{M}$ and δ are to be determined. $f(r) \in C_0^\infty(R)$, $0 \le f(r) \le 1$ and

$$f(r) = \begin{cases} 1, & |r| \le 1, \\ 0, & |r| \ge 2. \end{cases}$$

Consider the following problem

$$\begin{cases} w_t = h(w)w_{xx} + (m-1)g(w)(w_x)^2 + w^p, & 0<x<1, t>0, \\ w_x|_{x=0} = 0, \qquad w_x|_{x=1} = -|w|^{\alpha-1}w, & t \ge 0, \\ w|_{t=0} = u_0, & 0 \le x \le 1. \end{cases} \tag{3.1}$$

We first set

$$\delta = \delta_0 = \frac{1}{2} \min_{x\in[0,1]} u_0(x), \quad \overline{M} = \overline{M}_0 = 2M.$$

The standard parabolic equation theory ([4],Th.7.4) assumes the existence and uniqueness of

$$w_0(x,t) \in H^{2+\beta,1+\frac{\beta}{2}}(\overline{G}_T),$$

for some $\beta \in (0,1)$, solution of (3.1). By the continuity of $w_0(x,t)$, there is a $t_0 > 0$, such that

$$\delta_0 \le w_0 \le \overline{M}_0, \qquad \text{for } (x,t) \in \overline{G}_{t_0}.$$

Let

$$T_0 = \sup\left\{t_0 | \ \ \delta_0 \le w_0 \le \overline{M}_0, \ (x,t) \in \overline{G}_{t_0}\right\}.$$

Thus by the definition of $h(r)$ and $g(r)$, w_0 is a solution of (1.1) on $\overline{G}_{T_0}$, or

$$w_0 = u, \qquad \text{for } t \in [0, T_0]. \tag{3.2}$$

Moreover, $\lim\limits_{t\longrightarrow 0} \int_0^1 |u - u_0| dx = 0$.

Next, we set

$$\begin{aligned} \delta = \delta_1 &= \frac{1}{2} \min\{[\eta^{\frac{m}{q}} + C_T(1 + T_0^{-\frac{1}{2}})]^{\frac{q}{m}}, \delta_0\}, \\ \overline{M} = \overline{M}_1 &= 2\max(2M, C_0). \end{aligned}$$

Where,

$$\eta = [(\alpha + m - 2)T + \int_0^1 u_0^{2-m-\alpha} dx]^{\frac{1}{1-m-\alpha}}.$$

For δ_1 and $\overline{M}$, there also exists a unique solution of (3.1) $w_1(x,t) \in H^{2+\beta,1+\frac{\beta}{2}}(\overline{G}_T)$, and a point t_1 such that

$$\delta_1 \le w_1 \le \overline{M}_1, \qquad \text{for } (x,t) \in \overline{G}_{t_1}.$$

Let

$$T_1 = \sup\left\{t_1 | \ \ \delta_1 \le w_1 \le \overline{M}_1, \ (x,t) \in \overline{G}_{t_1}\right\}.$$

Thus w_1 is a solution of (1.1) on $\overline{G}_{T_1}$, or

$$w_1 = u, \qquad \text{for } t \in [0, T_1]. \tag{3.3}$$

Clearly, using the lemma 2 of [6], we know $T_0 \le T_1$.

We end this step by showing that $T_1 = T$. By the definitions of T_1, $\overline{M}_1$ and δ_1, there is a point $x_1 \in [0,1]$ such that

$$u(x_1, T_1) = \overline{M}_1, \tag{3.4}$$

or

$$u(x_1, T_1) = \delta_1. \tag{3.5}$$

If $T_1 < T$, then by lemma 1, we have

$$u(x, T_1) \le C_0, \qquad \text{for } x \in [0,1]. \tag{3.6}$$

Since $C_0 < \overline{M}_1$, so (3.4) contradicts (3.6). On the other hand, since $T_1 < T$, lemma 3 implies

$$\begin{aligned}
\int_0^1 u(x, T_1) dx &\ge [(\alpha + m - 2)T_1 + \int_0^1 u_0^{2-m-\alpha} dx]^{\frac{1}{1-m-\alpha}} \\
&> [(\alpha + m - 2)T + \int_0^1 u_0^{2-m-\alpha} dx]^{\frac{1}{1-m-\alpha}} \\
&= \eta.
\end{aligned}$$

Thus there is a $x_2 \in [0,1]$ such that $u(x_2, T_1) \ge \eta$. Using lemma 2 and its Corollary we have

$$\begin{aligned}
u^{\frac{m}{q}}(x, T_1) &\le u^{\frac{m}{q}}(x_2, T_1) + C_{T_1}(1 + {T_1}^{-\frac{1}{2}}) \\
&\le \eta^{\frac{m}{q}} + C_T(1 + {T_0}^{-\frac{1}{2}}).
\end{aligned}$$

Hence,

$$\begin{aligned}
u(x, T_1) &\ge [\eta^{\frac{m}{q}} + C_T(1 + {T_0}^{-\frac{1}{2}})]^{\frac{q}{m}} \\
&\ge 2\delta_1, \qquad \text{for } x \in [0,1].
\end{aligned} \tag{3.7}$$

Clearly, (3.7) contradicts (3.5). Thus, $T_1 = T$ and

$$w_1 = u(x,t), \qquad \text{for } (x,t) \in G_T.$$

Therefore, $u(x,t)$ is a solution of (1.1) on G_T. The bootstrap argument ([5]) shows that $u \in C^\infty(G_T)$. Recalling from the arbitrariness of T, we know that this step is complete.

STEP 2 Assume u_0 be as (1.2). We will prove that the conclusions of the theorem are valid.

For $0 < \delta < \frac{1}{12}$, let

$$u_0^* = \begin{cases} u_0, & x \in [2\delta, 1-2\delta], \\ 0, & x \overline{\in} [2\delta, 1-2\delta], \end{cases}$$

and

$$u_{0\delta} = \delta + \delta^\alpha x^2(1-x) + \int_0^1 u_0^*(y) J(x - \delta y) dy.$$

Where, J is a smooth averaging kernel. Clearly, $u_{0\delta}$ satisfies the conditions of STEP 1 and

$$\lim_{\delta\longrightarrow 0}\|u_{0\delta}-u_0\|_{L^1(0,1)}=0.$$

For any given $T>0$, we consider the problem

$$\begin{cases} u_t=(u^{m-1}u_x)_x+u^p, & 0<x<1, 0<t\le T,\\ u_x|_{x=0}=0, \qquad u_x|_{x=1}=-u^\alpha, & 0\le t\le T,\\ u|_{t=0}=u_{0\delta}, & 0\le x\le 1.\end{cases}$$

STEP 1 assures that there is a smooth solution $u_\delta\in C^\infty(G_T)\bigcap C([0,T];L^1(0,1))$ and

$$\int_0^1 u_\delta(x,t)dx\ge\int_0^1 u_{0\delta}(x)dx-\int_0^t u_\delta^{m-1+\alpha}(1,\tau)d\tau, \qquad \text{for } t\in[0,T]. \tag{3.8}$$

Recalling from $\int_0^1 u_{0\delta}dx\longrightarrow\overline{u}_0$ as $\delta\longrightarrow 0$, hence we know that there are δ_0 and t_0 such that

$$\int_0^1 u_\delta(x,t)dx\ge\frac{1}{2}\overline{u}_0, \qquad \text{for } \delta\in(0,\delta_0),\ t\in[0,2t_0]. \tag{3.9}$$

For any given $\tau\in(0,2t_0]$, lemma 1 and lemma 2 and Arzela's theorem assure the existence of subsequence $\{u_{\delta_k}(x,t)\}$ and a function $u^*(x,t)$ such that

$$\lim_{\delta_k\longrightarrow 0}u_{\delta_k}(x,t)=u^*(x,t), \qquad \text{uniformly on } x\in[0,1] \tag{3.10}$$

for $t\in[\tau,2t_0]$. On the other hand, (3.9) implies that for any $\delta\in(0,\delta_0)$, there is a point (x_3,t) such that

$$u_\delta(x_3,t)\ge\frac{1}{2}\overline{u}_0, \qquad \text{for } t\in[\tau,2t_0]. \tag{3.11}$$

By lemma 2,

$$\begin{aligned} u_\delta^{\frac{m}{q}}(x,t) &\le u_\delta^{\frac{m}{q}}(x_3,t)+C_T(1+t^{-\frac{1}{2}})\\ &\le u_\delta^{\frac{m}{q}}(x_3,t)+C_T(1+\tau^{-\frac{1}{2}}), \qquad \text{for } (x,t)\in[0,1]\times[\tau,2t_0].\end{aligned}$$

Using (3.11),

$$\begin{aligned} u_\delta(x,t) &\ge [(\frac{\overline{u}_0}{2})^{\frac{m}{q}}+C_T(1+\tau^{-\frac{1}{2}})]^{\frac{q}{m}}\\ &> 0, \qquad \text{for } (x,t)\in[0,1]\times[\tau,2t_0]. \end{aligned} \tag{3.12}$$

Set

$$\begin{aligned} A &= u_\delta^{m-1}, \\ B &= \frac{4(m-1)}{m^2}((u_\delta^{\frac{m}{2}})_x)^2 + u_\delta^p. \end{aligned}$$

Thus lemma 2 and (3.12) imply that there is a positive constant μ which doesn't depend on $\delta \in (0, \delta_0)$ such that

$$0 < A < \mu, \qquad |B| < \mu, \qquad\qquad \text{for } (x,t) \in [0,1] \times [\tau, 2t_0].$$

Notice that u_δ satisfies the linear equation

$$\frac{\partial}{\partial t} u_\delta = A \frac{\partial^2}{\partial x^2} u_\delta + B.$$

For any $\varepsilon \in (0, \frac{1}{2})$, [3](p.104) shows that there are positive constants h,ν and C, which don't depend on $\delta \in (0, \delta_0)$, such that

$$|u_\delta(x,t_2) - u_\delta(x,t_1)| \le C|t_2 - t_1|^h,$$

for $t_1,\ t_2 \in [\tau, 2t_0]$, $|t_1 - t_2| < \nu, x \in [\varepsilon, 1-\varepsilon]$. Certainly, we also have $|u_{\delta_k}(x,t_2) - u_{\delta_k}(x,t_1)| \le C|t_2 - t_1|^h$. Letting $\delta_k \longrightarrow 0$ yields

$$|u^*(x,t_2) - u^*(x,t_1)| \le C|t_2 - t_1|^h,$$

for $t_1,\ t_2 \in [\tau, 2t_0]$, $|t_1 - t_2| < \nu, x \in [\varepsilon, 1-\varepsilon]$. Thus, for any given $x \in (0,1)$, $u^*(x,t)$ is continuous with respect to $t \in [\tau, 2t_0]$. On the other hand, lemma 2 implies that there is a positive constant K such that $|u_{\delta_k}| \le K$ on $(x,t) \in [\tau, 2t_0] \times [0,1]$, so $|u_{\delta_k}(x_2,t) - u_{\delta_k}(x_1,t)| \le K|x_2 - x_1|$. Letting $\delta_k \longrightarrow 0$, we have

$$|u^*(x_2,t) - u^*(x_1,t)| \le K|x_2 - x_1| \qquad \text{uniformly on } x_1, x_2 \in [0,1], t \in [\tau, 2t_0].$$

Now we have $u^* \in C([0,1] \times [\tau, 2t_0])$ and $u^*(x,t) > 0$ for $(x,t) \in [0,1] \times [\tau, 2t_0]$. By lemma 5 of [2], we know that u^* satisfies the equation and the boundary conditions of (1.1). Clearly, $u^* \in C([\tau, 2t_0];\ L^1(0,1))$. Because $\tau > 0$ is arbitrary, so $u^* \in C((0, 2t_0]; L^1(0,1))$.

To show that u^* is a solution of (1.1) on G_{2t_0}, we want to prove $\|u^* - u_0\|_{L^1(0,1)} \longrightarrow 0$ as $t \longrightarrow 0$.

For any δ_k, δ_{k+j}, lemma 4 implies

$$\begin{aligned} \|u_{\delta_k} - u_{\delta_{k+j}}\|_{L^1(0,1)} &\le \|u_{0\delta_k} - u_{0\delta_{k+j}}\|_{L^1(0,1)} + \int_0^t \|u_{\delta_k}^p - u_{\delta_{k+j}}^p\|_{L^1(0,1)} d\tau \\ &\le \|u_{0\delta_k} - u_0\|_{L^1(0,1)} + \|u_{0_{\delta_{k+j}}} - u_0\|_{L^1(0,1)} \\ &\quad + \int_0^t \|u_{\delta_k}^p - u_{\delta_{k+j}}^p\|_{L^1(0,1)} d\tau, \qquad \text{for } t \in (0, 2t_0]. \end{aligned}$$

Letting $j \longrightarrow \infty$ yields

$$\|u_{\delta_k} - u^*\|_{L^1(0,1)} \leq \|u_{0\delta_k} - u_0\|_{L^1(0,1)} + \int_0^t \|u_{\delta_k}^p - u^{*p}\|_{L^1(0,1)} d\tau, \qquad \text{for } t \in (0, 2t_0].$$

Notice that

$$\begin{aligned} \|u^* - u_0\|_{L^1(0,1)} &\leq \|u^* - u_{\delta_k}\|_{L^1(0,1)} + \|u_{\delta_k} - u_{0\delta_k}\|_{L^1(0,1)} + \|u_{0\delta_k} - u_0\|_{L^1(0,1)} \\ &\leq 2\|u_{0\delta_k} - u_0\|_{L^1(0,1)} + \|u_{\delta_k} - u_{0\delta_k}\|_{L^1(0,1)} \\ &\quad + \int_0^t \|u_{\delta_k}^p - u^{*p}\|_{L^1(0,1)} d\tau, \qquad \text{for } t \in (0, 2t_0]. \end{aligned}$$

Thus,

$$\lim_{t \longrightarrow 0} \|u^* - u_0\|_{L^1(0,1)} \leq 2\|u_{0\delta_k} - u_0\|_{L^1(0,1)}.$$

Letting $\delta_k \longrightarrow 0$ shows

$$\lim_{t \longrightarrow 0} \|u^* - u_0\|_{L^1(0,1)} = 0.$$

Next, we consider the problem

$$\begin{cases} u_t = (u^{m-1}u_x)_x + u^p, & 0 < x < 1, \quad t_0 < t \leq T, \\ u_x|_{x=0} = 0, \qquad u_x|_{x=1} = -u^\alpha, & t_0 \leq t \leq T, \\ u|_{t=t_0} = u^*(x, t_0), & 0 \leq x \leq 1. \end{cases} \tag{3.13}$$

Since $u^*(x, t_0) > 0$ and $u^*(x, t_0)$ is smooth enough for $x \in [0, 1]$, the conclusion of STEP 1 shows that there is a function u^{**} to solve (3.13). Now we define a function

$$u(x, t) = \begin{cases} u^*, & t \in [0, t_0], \\ u^{**}, & t \in [t_0, T]. \end{cases}$$

Clearly, u is a solution of (1.1) in G_T and the bootstrap argument ([5]) shows $u \in C^\infty(G_T)$.

To end the proof of our theorem, we assume

$$\overline{u}(x, t) = \begin{cases} u_{11}, & t \in [0, t_*], \\ u_{12}, & t \in [t_*, T], \end{cases}$$

$$\overline{\overline{u}}(x, t) = \begin{cases} u_{21}, & t \in [0, t_*], \\ u_{22}, & t \in [t_*, T], \end{cases}$$

in which, $\overline{u}$ and $\overline{\overline{u}}$ are two solutions corresponding to initial values u_{10} and u_{20}. Thus lemma 4 shows

$$\|u_{21} - u_{11}\|_{L^1(0,1)} \leq \|u_{20} - u_{10}\|_{L^1(0,1)} + \int_0^t \|u_{21}^p - u_{11}^p\|_{L^1(0,1)} d\tau, \qquad \text{for } t \in [0, t_*].$$

By lemma 1, u_{ij} are bounded on G_T for $i, j = 1, 2$. Hence we can set t_* small enough such that

$$\|u_{21} - u_{11}\|_{L^1(0,1)} \leq 2\|u_{20} - u_{10}\|_{L^1(0,1)}, \qquad \text{for } t \in [0, t_*]. \quad (3.14)$$

Notice that (2.7) yields

$$\begin{aligned} \frac{d}{dt}\int_0^1 |u_{22} - u_{12}|dx &\leq \int_0^1 |u_{22}^p - u_{12}^p|dx \\ &\leq p\xi^{p-1}\int_0^1 |u_{22} - u_{12}|_{L^1(0,1)}dx, \qquad \text{for } t \in [t_*, T], \end{aligned}$$

in which,

$$\begin{aligned} \xi &= \min_{(x,t)\in[0,1]\times[t_*,T]}(u_{12}, u_{22}) \\ &> 0. \end{aligned}$$

Using (3.14) we have

$$\begin{aligned} \|u_{22} - u_{12}\|_{L^1(0,1)} &\leq (\|u_{22} - u_{12}\|_{L^1(0,1)})_{t=t_*}e^{p\xi^{p-1}t} \\ &\leq 2\|u_{20} - u_{10}\|_{L^1(0,1)}e^{p\xi^{p-1}t}, \qquad \text{for } t \in [t_*, T]. \end{aligned} \quad (3.15)$$

It follows from $0 < p < 1$ that $e^{p\xi^{p-1}t} \leq 1$. Combining (3.14) and (3.15) yields (1.3), the uniqueness of the solution is followed immediately.

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